

UNORIENTED BAND SURGERY ON KNOTS AND LINKS

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ABSTRACT. We consider a relation between two kinds of unknotting numbers defined by using a band surgery on unoriented knots; the band-unknotting number and $H(2)$ -unknotting number, which we may characterize in terms of the first Betti number of surfaces in S^3 spanning the knot and the trivial knot. We also give several examples for these numbers.

1. INTRODUCTION

A band surgery is an operation which deforms a link into another link. Let L be a link and $b : I \times I \rightarrow S^3$ an embedding such that $L \cap b(I \times I) = b(I \times \partial I)$, where I is the unit interval $[0, 1]$. Then we may obtain a new link $M = (L \setminus b(I \times \partial I)) \cup b(\partial I \times I)$, which is called a link obtained from L by the *band surgery* along the band B , where $B = b(I \times I)$; see Fig. 1.

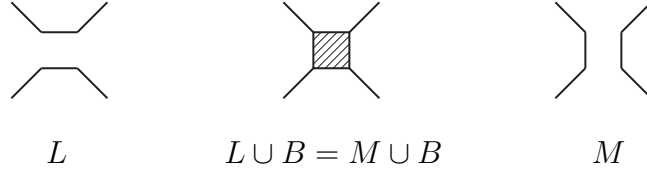


FIGURE 1. M is obtained from L by a band surgery along the band B , and vice versa.

A band surgery appears in various aspects in knot theory. For example, it is an essential tool in the study of surfaces embedded in 4-space; it is motivated by the study of DNA site-specific recombinations; cf. [6, 31]. In this paper, we study an unoriented band surgery, that is, we consider a band surgery for unoriented knots and links. Using an unoriented band surgery, we can define two numerical invariants for a knot. One is the *band-unknotting number* of a knot K , $u_b(K)$, which is the minimal number of band surgeries to deform K into the trivial knot. The other is the *$H(2)$ -unknotting number* of K , $u_2(K)$, introduced by Hoste, Nakanishi, and Taniyama [11], which is the minimal number of component-preserving band surgeries to deform K into the trivial knot, and so the unknotting sequence which realizes

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the $H(2)$ -unknotting number consists of only knots, whereas the unknotting sequence which realizes the band-unknotting number may contain a link with more than one component. Then by definition $u_b(K) \leq u_2(K)$. Moreover, we have:

Corollary 3.2. *For a knot K ,*

$$u_b(K) = u_2(K) - 1 \text{ or } u_2(K).$$

Furthermore, if $u_b(K)$ is odd, then $u_b(K) = u_2(K)$; equivalently, if $u_2(K)$ is either one or even, then $u_b(K) = u_2(K)$.

Actually, we have $u_b(8_{18}) = 2$ and $u_2(8_{18}) = 3$ (Example 6.5(i)), which may be generalized as follows:

Theorem 7.1. *There exist infinitely many knots K such that $u_b(K) = 2$ and $u_2(K) = 3$.*

On the other hand, Taniyama and Yasuhara [32, Theorem 5.1] characterized the band-unknotting number $u_b(K)$ as $\min_F \{\beta_1(F)\} - 1$, where the minimum is taken over all connected (orientable or nonorientable) surfaces F in S^3 spanning the knot K and the trivial knot, where $\beta_1(F)$ is the first Betti number of F . Similarly, we may characterize the $H(2)$ -unknotting number $u_2(K)$ as $\min_{F_N} \{\beta_1(F_N)\} - 1$, where the minimum is taken over all connected *nonorientable* surfaces F_N in S^3 spanning the knot K and the trivial knot (Theorem 4.2). In the following, we will consider the band-Gordian distance and $H(2)$ -Gordian distance, which generalize the band-unknotting number and $H(2)$ -unknotting number, respectively.

This paper is organized as follows: In Sec. 2, we give definitions of band-Gordian distance and band-unknotting number, and those of $H(2)$ -Gordian distance and $H(2)$ -unknotting number. In Sec. 3, we prove Theorem 3.1, which generalizes Corollary 3.2 to the distance. In Sec. 4, we give characterizations of the band-Gordian distance and $H(2)$ -Gordian distance in terms of the first Betti number of surfaces in S^3 spanning the knots (Eq. (4.1) and Theorem 4.2). In Sec. 5, we review some invariants; the homology groups of the cyclic branched covering space of a link, the Jones polynomial, the Arf invariant, the signature, and the Q polynomial. We use them to evaluate the band-Gordian distance. In Sec. 6, we give estimations for the band-Gordian distance and the band-unknotting number in terms of the usual Gordian distance and unknotting number, respectively (Theorem 6.1). Using Theorem 6.1 and invariants given in Sec. 5 we determine the band-Gordian distances and band-unknotting numbers for examples with $H(2)$ -Gordian distance or $H(2)$ -unknotting number 3 given in [16]. In Sec. 7, we prove Theorem 7.1. In Sec. 8, we give a complete table of band-unknotting numbers and $H(2)$ -unknotting numbers for knots with up to 9

crossings. In Sec. 9, we give a relation for a special value of the Jones polynomial of a slice knot.

2. BAND-GORDIAN DISTANCE AND $H(2)$ -GORDIAN DISTANCE

In this section, we introduce the band-Gordian distance and $H(2)$ -Gordian distance. It is easy to see that any connected unoriented link diagram can be deformed into the trivial knot diagram without any crossing by smoothing appropriately at every crossing as shown in Fig. 2; cf. [18, Part 1, Sec. 4]. Since a smoothing is realized by an unoriented band surgery, any link can be deformed into the trivial knot by a sequence of unoriented band surgeries. Thus any two links are related by a sequence of unoriented band surgeries.

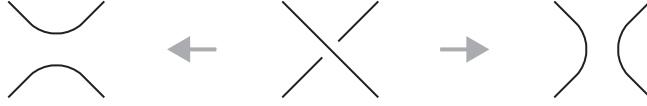


FIGURE 2. Smoothing a crossing.

Definition 2.1. Let L and M be unoriented links. The *band-Gordian distance* from L to M , denoted by $d_b(L, M)$, is defined to be the minimal number of unoriented band surgeries needed to deform L into M . In particular, the *band-unknotted number* of a knot K , $u_b(K)$, is defined to be the band-Gordian distance from K to the trivial knot U ; $u_b(K) = d_b(K, U)$.

Hoste, Nakanishi, and Taniyama [11] introduced an $H(n)$ -move, which is a deformation of a link diagram. In particular, an $H(2)$ -move is a band surgery which requires to keep the number of the components. An $H(2)$ -move is an unknotting operation, that is, any knot K can be deformed into the trivial knot by a sequence of $H(2)$ -moves [11, Theorem 1]. Let J and K be knots. The *$H(2)$ -Gordian distance* from J to K , denoted by $d_2(J, K)$, is defined to be the minimal number of $H(2)$ -moves needed to transform J into K , and the *$H(2)$ -unknotting number* of a knot K , denoted by $u_2(K)$, is defined to be the $H(2)$ -Gordian distance from K to the trivial knot U ; $u_2(K) = d_2(K, U)$.

3. BAND SURGERIES AND SURFACES IN S^3

For two knots J and K , by definition we have $d_b(J, K) \leq d_2(J, K)$. Moreover we have:

Theorem 3.1. *For any knots J and K , we have*

$$d_b(J, K) = d_2(J, K) - 1 \text{ or } d_2(J, K). \quad (3.1)$$

Furthermore, if $d_b(J, K)$ is odd, then $d_b(J, K) = d_2(J, K)$; equivalently, if $d_2(J, K)$ is either one or even, then $d_b(J, K) = d_2(J, K)$.

In particular, we have:

Corollary 3.2. *For a knot K ,*

$$u_b(K) = u_2(K) - 1 \text{ or } u_2(K). \quad (3.2)$$

Furthermore, if $u_b(K)$ is odd, then $u_b(K) = u_2(K)$; equivalently, if $u_2(K)$ is either one or even, then $u_b(K) = u_2(K)$.

We prove Theorem 3.1 using Lemma 3.4 below, which allows us to understand the band-Gordian distance $d_b(J, K)$ in terms of the first Betti number of a surface in S^3 bounding a 2-component link whose components are isotopic to J and K . Although Lemma 3.4 is essentially given by Taniyama and Yasuhara [32, Theorem 5.1], for the sake of completeness we provide a proof.

We may perform a band surgery along a several number of disjoint bands simultaneously. Let L be a link and $b_i : I \times I \rightarrow S^3$ ($i = 1, \dots, n$) are disjoint embeddings, that is, $b_i(I \times I) \cap b_j(I \times I) = \emptyset$ for $i \neq j$ such that $L \cap b_i(I \times I) = b_i(I \times \partial I)$. Then we obtain another link M ;

$$M = \left(L \setminus \bigcup_{i=1}^n b_i(I \times \partial I) \right) \cup \bigcup_{i=1}^n b_i(\partial I \times I). \quad (3.3)$$

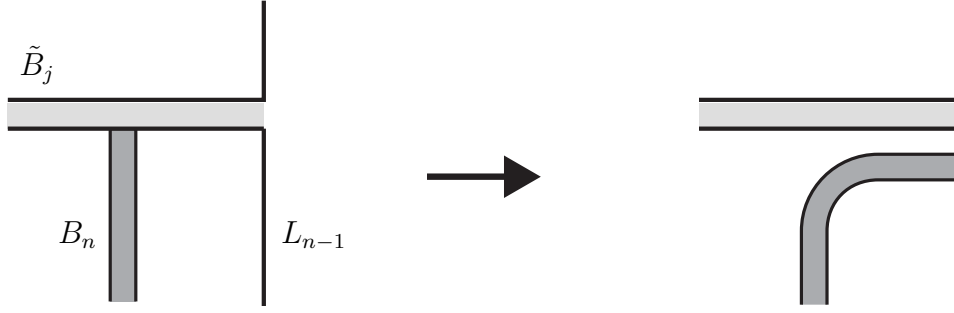
Then M is called the link obtained from L by the (multiple) band surgery along the bands B_1, \dots, B_n , where $B_i = b_i(I \times I)$, which we denote by $h(L; B_1, \dots, B_n)$.

In order to give a normal form for an embedded surface in S^4 the following lemma is essentially proved in [19, Lemma 1.14] and [14, Lemma 2.4].

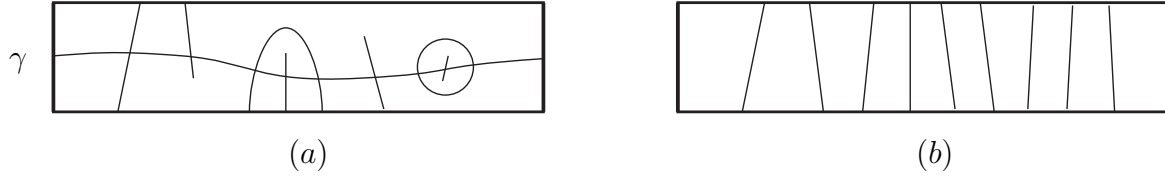
Lemma 3.3. *Suppose that L and M are links, which are related by a sequence of n band surgeries. Then there exist mutually disjoint bands B_1, \dots, B_n such that $M = h(L; B_1, \dots, B_n)$.*

Proof. We use induction on n . The case $n = 1$ is trivial. Let L_i , $i = 0, 1, \dots, n$, be links such that $L_0 = L$, $L_n = M$, and L_i is obtained from L_{i-1} by a band surgery along b_i ; $L_i = h(L_{i-1}, B_i)$, $1 \leq i \leq n$, where $B_j = b_j(I \times I)$. Suppose that the lemma holds for $n - 1$. Then there exist mutually disjoint bands $\tilde{B}_1, \dots, \tilde{B}_{n-1}$ such that $L_{n-1} = h(L; \tilde{B}_1, \dots, \tilde{B}_{n-1})$. Let α be an attaching arc of B_n ; $\alpha = b_n(I \times \{t\})$, $t = 0, 1$. If $\alpha \cap \tilde{B}_j \neq \emptyset$, $j < n$, we slide B_n along L_{n-1} as shown in Fig. 3. Therefore we may assume that $\alpha \cap (\tilde{B}_1 \cup \dots \cup \tilde{B}_{n-1}) = \emptyset$.

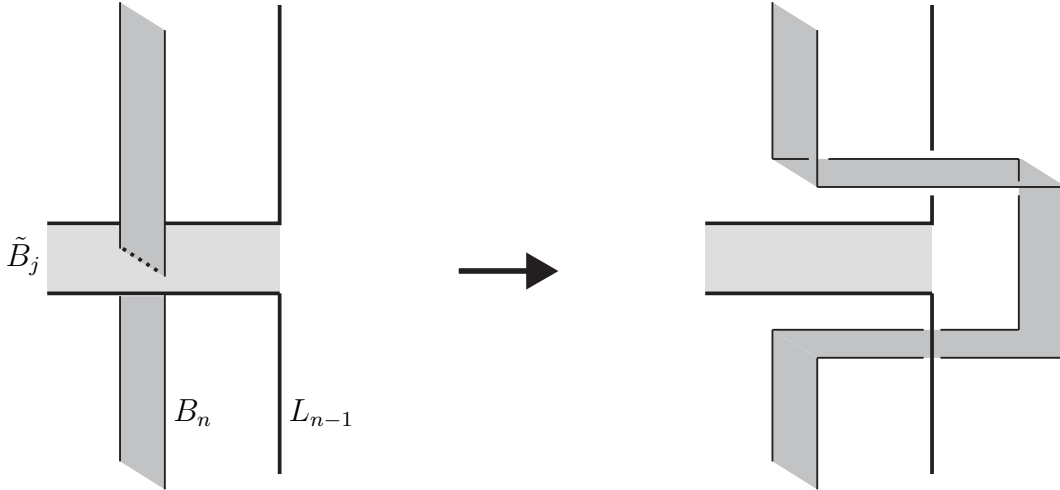
Since we may assume that B_n and $\tilde{B}_1 \cup \dots \cup \tilde{B}_{n-1}$ intersect transversely, the intersection $B_n \cap (\tilde{B}_1 \cup \dots \cup \tilde{B}_{n-1})$ is a 1-manifold, that is, simple loops and arcs. Now we choose a proper simple arc γ in B_n connecting the two attaching arcs of B_n as in Fig. 4(a). Replace B_n with a sufficiently small regular neighborhood of γ in B_n . Then we may assume that the

FIGURE 3. Sliding the band B_n along L_{n-1} .

intersection $B_n \cap (\tilde{B}_1 \cup \cdots \cup \tilde{B}_{n-1})$ consists of proper simple arcs in B_n as in Fig. 4(b). In particular, $B_n \cap (\partial\tilde{B}_1 \cup \cdots \cup \partial\tilde{B}_{n-1}) = \emptyset$.

FIGURE 4. Repalacing the band B_n .

Finally, we deform the band B_n so that $B_n \cap (\tilde{B}_1 \cup \cdots \cup \tilde{B}_{n-1}) = \emptyset$ as in Fig. 5. Let \tilde{B}_n be the resulting band. Then the bands $\tilde{B}_1, \dots, \tilde{B}_n$ are mutually disjoint and $M = h(L; \tilde{B}_1, \dots, \tilde{B}_n)$. Therefore the lemma holds for n , completing the proof. \square

FIGURE 5. Deforming the band B_n .

Lemma 3.4. *If two knots J and K are related by a sequence of n band surgeries, then there exists a compact connected surface F in S^3 such that:*

(i) F spans the two knots J and K , that is, ∂F is a 2-component link $\tilde{J} \cup \tilde{K}$ with \tilde{J} and \tilde{K} isotopic to J and K , respectively; and

(ii) $\beta_1(F) = n + 1$,

where $\beta_1(F)$ is the first Betti number of F .

Proof. By Lemma 3.3 there exist mutually disjoint bands B_1, \dots, B_n such that $K = h(J; B_1, \dots, B_n)$. We take an annulus A embedded in S^3 so that one of the boundary components is J and $A \cap B_i = J \cap B_i$ is the attaching arcs of B_i for each i . Let F be the surface $A \cup (\bigcup_{i=1}^n B_i)$. Then F satisfies the conditions (i) and (ii). \square

Proof of Theorem 3.1. By definition, we have $d_b(J, K) \leq d_2(J, K)$. Let $n = d_b(J, K)$. By Lemma 3.4, there exists a compact connected surface F such that $\partial F = \tilde{J} \cup \tilde{K}$ and $\beta_1(F) = n + 1$, where \tilde{J} and \tilde{K} are ambient isotopic to J and K , respectively. There are two cases.

(i) F is nonorientable. We can deform F into the surface as shown in Fig. 6(a), which is constructed by attaching n half-twisted bands and 1 bands to a disc; cf. [4]. In fact, F is the connected sum of n projective planes minus two open discs, which is homeomorphic to the surface as shown in Fig. 6(b). Then adding n bands to \tilde{J} as shown in Fig. 6(c), we obtain a parallel link which bounds an annulus whose center line is isotopic to K . This implies that the knot J is transformed into K by performing the $H(2)$ -move n times. Thus $d_2(J, K) \leq n$, and so we have $d_b(J, K) = d_2(J, K)$.

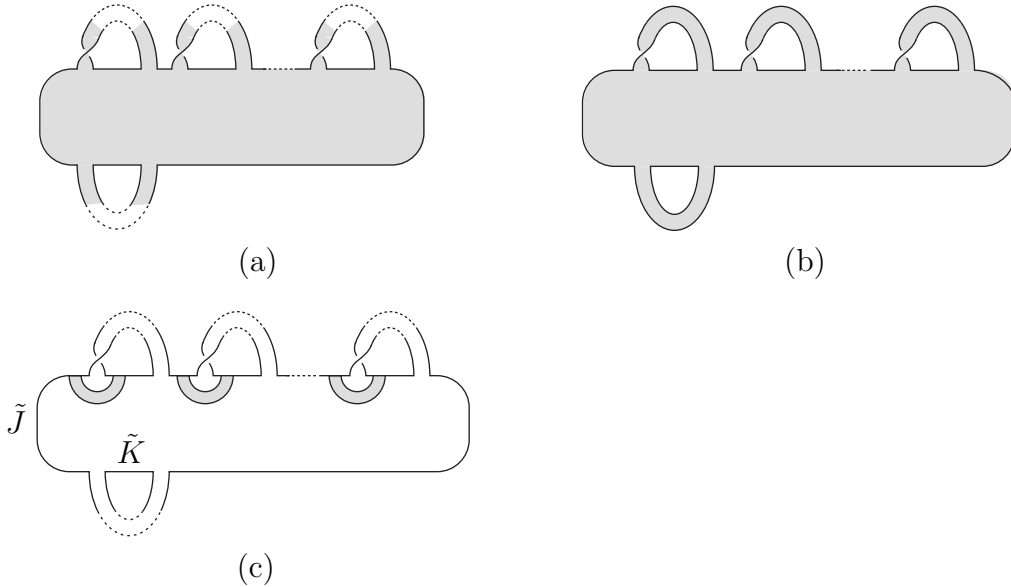


FIGURE 6. The case F is nonorientable.

(ii) F is orientable. Then F is a Seifert surface for the link $\tilde{J} \cup \tilde{K}$ and is represented in the disc-band form as shown in Fig. 7(a), which is constructed by attaching $(n + 1)$ bands to a

disc, and so n is an even number and the genus of F is $n/2$; cf. [33], Attaching a half-twisted band to F , we obtain a nonorientable surface F' as shown in Fig. 7(b). Then $\beta_1(F') = n + 2$ and the boundary of F' is isotopic to the link $\tilde{J} \cup \tilde{K}$. Therefore, from Case (i) we have $d_2(J, K) \leq n + 1$, giving Eq. (3.1). Noting that n is even, the proof is complete. \square

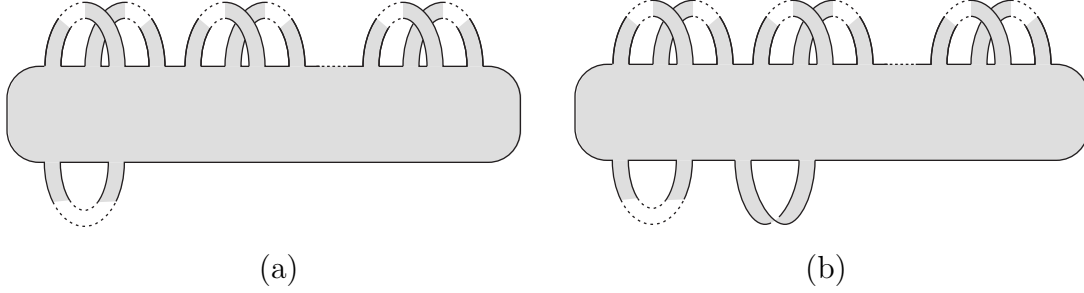


FIGURE 7. The case F is orientable.

4. CHARACTERIZATIONS OF THE BAND-GORDIAN DISTANCE AND $H(2)$ -GORDIAN DISTANCE

For two knots J and K , Taniyama and Yasuhara [32, Sec. 5] defined two indices.

- $\tilde{d}_C(J, K) = \min_F \{\beta_1(F)\} - 1$, where the minimum is taken over all connected (orientable or nonorientable) surfaces F in S^3 spanning the two knots J and K .
- $\tilde{c}(J, K)$, the minimum number of critical points of a locally flat (orientable or nonorientable) surface in $S^3 \times [0, 1]$ bounded by $J \times \{0\}$ and $K \times \{1\}$.

Then they showed [32, Theorem 5.1]:

$$d_b(J, K) = \tilde{d}_C(J, K) = \tilde{c}(J, K). \quad (4.1)$$

In fact, from the proof of Theorem 3.1 we see the first equality of Eq. (4.1).

Remark 4.1. In [32], the symbol $d_2(J, K)$ is used to mean the band-Gordian distance $d_b(J, K)$, but not our $H(2)$ -Gordian distance.

We define two analogous indices:

- $\tilde{d}_{C_N}(J, K) = \min_{F_N} \{\beta_1(F_N)\} - 1$, where the minimum is taken over all connected *nonorientable* surfaces F_N in S^3 spanning two knots J and K .
- $\tilde{c}_N(J, K)$, the minimum number of critical points of a locally flat *nonorientable* surface in $S^3 \times [0, 1]$ bounded by $J \times \{0\}$ and $K \times \{1\}$.

Then we have:

Theorem 4.2.

$$d_2(J, K) = \tilde{d}_{C_N}(J, K) = \tilde{c}_N(J, K). \quad (4.2)$$

Proof. We only prove the first equality. Let $d_2(J, K) = n$. By Lemma 3.4 there exists a nonorientable surface F spanning two knots J and K with $\beta_1(F) = n + 1$. In fact, let K_0, K_1, \dots, K_n be a sequence of knots such that $K_0 = J$, $K_n = K$, and K_i is obtained from K_{i-1} by a single $H(2)$ -move. Since K_1 is obtained from J by adding a half-twisted band, the surface F constructed as in the proof of Lemma 3.4 is nonorientable. Therefore, $\tilde{d}_{C_N}(J, K) \leq n$.

Conversely, let $n = \tilde{d}_{C_N}(J, K)$, and let F be a nonorientable surface spanning the two knots J and K with $\beta_1(F) = n + 1$. Then F may be illustrated as in Fig. 6(a). By adding n bands as in Fig. 6(c), J is transformed into K , and so we have $d_2(J, K) \leq n$, completing the proof. \square

5. SOME INVARIANTS

In this section, we review the definitions and some properties of the homology groups of the cyclic branched covering space, Jones polynomial, Arf invariant, signature, and Q polynomial, which enable us to estimate the band-Gordian distance and band-unknotting number.

5.1. The homology group of the cyclic branched covering space. For an oriented link L , let $\Sigma_p(L)$ be the p -fold cyclic covering space of S^3 branched along L , and $e_p(L)$ the minimum number of generators of $H_1(\Sigma_p(L); \mathbf{Z})$. If either $p = 2$ or L is a knot, then the space $\Sigma_p(L)$ is irrelevant to the orientations of L . Therefore, we can prove the following in a similar way to Theorem 4 in [11].

Lemma 5.1. (i) *If two links L and M are related by a single unoriented band surgery, then $|e_2(L) - e_2(M)| \leq 1$.*

(ii) *If two knots J and K are related by a single $H(2)$ -move, then $|e_p(J) - e_p(K)| \leq p - 1$.*

This immediately implies:

Proposition 5.2. (i) *For two links L and M we have:*

$$d_b(L, M) \geq |e_2(L) - e_2(M)|; \quad (5.1)$$

$$u_b(L) \geq e_2(L). \quad (5.2)$$

(ii) *For two knots J and K we have:*

$$d_2(J, K) \geq |e_p(J) - e_p(K)| / (p - 1). \quad (5.3)$$

Remark 5.3. Proposition 5.2(i) gives the same statement as Proposition 5.2 with $p = 2$ in [11], which does not seem to hold for $p > 2$. Proposition 5.2(ii) with J unknot is given in Theorem 2.1 in [17].

Example 5.4. For each positive integer i let J_i be either the right-hand trefoil knot 3_1 or the left-hand trefoil knot $3_1^!$, and K_n the connected sum $J_1 \# \cdots \# J_n$. Then since $H_1(\Sigma_2(J_i); \mathbf{Z})$ is isomorphic to \mathbf{Z}_3 , $H_1(\Sigma_2(K_n); \mathbf{Z})$ is the n -fold direct sum of \mathbf{Z}_3 . Thus by Proposition 5.2 we have $d_b(K_m, K_n) \geq |m - n|$ and $u_b(K_n) \geq n$. Since $u_2(3_1) = 1$, we obtain

$$d_b(K_m, K_n) = |m - n|; \quad (5.4)$$

$$u_b(K_n) = n. \quad (5.5)$$

5.2. The Jones polynomial. The *Jones polynomial* $V(L; t) \in \mathbf{Z}[t^{-1/2}, t^{1/2}]$ [12] of an oriented link L is an isotopy invariant of an oriented link defined by the following formulas:

$$V(U; t) = 1; \quad (5.6)$$

$$t^{-1}V(L_+; t) - tV(L_-; t) = (t^{1/2} - t^{-1/2})V(L_0; t); \quad (5.7)$$

where U is the unknot and L_+ , L_- , L_0 are three links that are identical except near one point where they are as in Figure 8. We call the triple (L_+, L_-, L_0) a *skein triple*.

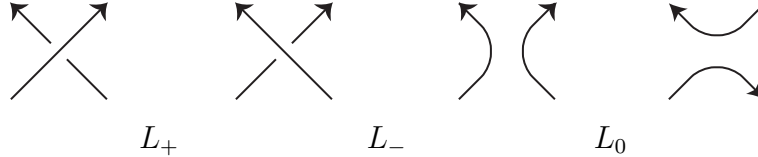


FIGURE 8. A skein triple.

Let L be a link with c components and $\delta(L) = \dim H_1(\Sigma_2(L); \mathbf{Z}_3)$. Then Lickorish and Millett [22, Theorem 3] have shown

$$V(L; \omega) = \pm i^{c-1} (i\sqrt{3})^{\delta(L)}, \quad (5.8)$$

where $\omega = e^{i\pi/3}$ and $V(L; \omega)$ means the value of $V(L; t)$ at $t^{1/2} = e^{i\pi/6}$. Notice that if L' is a link obtained from L by changing the orientation of one component of L , say K , then

$$V(L'; t) = t^{-3\lambda} V(L; t), \quad (5.9)$$

where λ is the linking number of K and the remainder of L ; cf. [27, Theorem 11.2.9]. Thus, $V(L; \omega) = \pm V(L'; \omega)$. This special value of the Jones polynomial evaluates the band-Gordian distance and band-unknotting number.

Theorem 5.5. *If two links L and M are related by a single band surgery, then*

$$|V(L; \omega)/V(M; \omega)| \in \{1, \sqrt{3}^{\pm 1}\}, \quad (5.10)$$

where the orientations of L and M are irrelevant.

This immediately implies the following:

Corollary 5.6. *Let L and M be links.*

- (i) *It holds that $d_b(L, M) \geq |\delta(L) - \delta(M)|$. In particular, $d_b(L) \geq \delta(L)$.*
- (ii) *If $|V(L; \omega)/V(M; \omega)| = \sqrt{3}^n$, then $d_b(L, M) \geq |n|$. In particular, if $|V(L; \omega)| = \sqrt{3}^n$, then $u_b(L) \geq n$.*

In order to prove Theorem 5.5 we use the following, which is due to Miyazawa [23, Proposition 4.2]; cf. [15, Lemma 2.1].

Lemma 5.7. *Let (L_+, L_-, L_0) be a skein triple. Then*

$$V(L_+; \omega)/V(L_-; \omega) \in \{\pm 1, i\sqrt{3}^{\pm 1}\}.$$

Proof of Theorem 5.5. If L and M are related by a single orientable band surgery, then by Theorem 2.2 in [15] we have

$$V(L; \omega)/V(M; \omega) \in \{\pm i, -\sqrt{3}^{\pm 1}\}. \quad (5.11)$$

We consider the case L and M are related by a nonorientable band surgery. Let $c(L)$ denote the number of components of a link L . For a skein triple (L_+, L_-, L_0) , let L_∞ be an oriented link represented by one of the diagrams in Figure 9, where (i) $c(L_+) < c(L_0)$ and (ii) $c(L_+) > c(L_0)$. Then we may assume L_0 and L_∞ are isotopic to L and M , respectively.

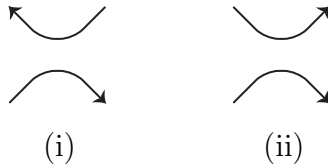


FIGURE 9. Two possible orientations for L_∞ .

Then we have the ‘ V_∞ ’ formuals [2]; cf. [20]:

$$V(L_+; t) - tV(L_-; t) + t^{3\lambda}(t-1)V(L_\infty; t) = 0; \quad (5.12)$$

$$V(L_+; t) - tV(L_-; t) + t^{3(\nu-\frac{1}{2})}(t-1)V(L_\infty; t) = 0, \quad (5.13)$$

where Eq. (5.12) holds for Case (i) and λ is the linking number of the right-hand component of L_0 in Figure 8 with the remainder of L_0 , and Eq. (5.13) holds for Case (ii) and ν is

the linking number of the bottom-right to top-left component of L_+ in Figure 8 with the remainder of L_+ . From Eqs. (5.7), (5.12) and (5.13) we have

$$\frac{V(L_0; t)}{V(L_\infty; t)} = \begin{cases} \frac{t^{-1/2}V(L_+; t) - t^{3/2}V(L_-; t)}{-t^{-3\lambda}(V(L_+; t) - tV(L_-; t))} & \text{for Case (i);} \\ \frac{t^{-1/2}V(L_+; t) - t^{3/2}V(L_-; t)}{-t^{-3\nu+3/2}(V(L_+; t) - tV(L_-; t))} & \text{for Case (ii).} \end{cases} \quad (5.14)$$

Then letting $x = V(L_+; \omega)/V(L_-; \omega)$, we have

$$\frac{V(L_0; \omega)}{V(L_\infty; \omega)} = \begin{cases} \frac{\omega^{-1/2}x - \omega^{3/2}}{-\omega^{-3\lambda}(x - \omega)} = \pm \frac{x - \omega^2}{\omega^{1/2}(x - \omega)} & \text{for Case (i);} \\ \frac{\omega^{-1/2}x - \omega^{3/2}}{-\omega^{-3(\nu-1/2)}(x - \omega)} = \pm i \frac{x - \omega^2}{\omega^{1/2}(x - \omega)} & \text{for Case (ii).} \end{cases} \quad (5.15)$$

For $x = 1, -1, -i\sqrt{3}, -i\sqrt{3}^{-1}$, we obtain $(x - \omega^2)/\omega^{1/2}(x - \omega) = \sqrt{3}, \sqrt{3}^{-1}, -i, i$, respectively. Thus by Lemma 5.7 the proof is complete. \square

Example 5.8. We use the same notation as in Example 5.4. Since $V(3_1; \omega) = -i\sqrt{3}$ and $V(3_1!; \omega) = i\sqrt{3}$, we have $|V(K_n; \omega)| = \sqrt{3}^n$, and so Corollary 5.6(ii) also implies Eqs. (5.4) and (5.5).

5.3. The Arf invariant. Using a certain quadratic form, Robertello [28] introduced a knot concordance invariant, the *Arf invariant* of a knot K , $\text{Arf}(K) \in \mathbf{Z}_2$. Using the Jones polynomial, one can define this invariant as follows:

$$V(K; i) = (-1)^{\text{Arf}(K)}; \quad (5.16)$$

see [24]. By this formula, we can calculate the Arf invariant recursively.

5.4. The signature. Murasugi [25] introduced the *signature* of an oriented link L , $\sigma(L) \in \mathbf{Z}$. For a skein triple (L_+, L_-, L_0) , Murasugi showed the following; see [25, Lemma 7.1] and [26, Theorem 1]:

$$|\sigma(L_\pm) - \sigma(L_0)| \leq 1. \quad (5.17)$$

Moreover, Giller [8] (cf. [27, Theorem 6.4.7]) found that the signature of a knot can be determined by the following three axioms:

- (i) For the trivial knot U , $\sigma(U) = 0$.
- (ii) Let (L_+, L_-, L_0) be a skein triple. If L_\pm are knots, then

$$\sigma(L_-) - 2 \leq \sigma(L_+) \leq \sigma(L_-). \quad (5.18)$$

- (iii) For a knot K , let $\text{sign}V(K; -1) = V(K; -1)/|V(K; -1)|$. Then

$$(-1)^{\sigma(K)/2} = \text{sign}V(K; -1). \quad (5.19)$$

Note that the signature of a knot K , $\sigma(K)$, is an even integer.

5.5. The Q polynomial. The Q polynomial $Q(L; z) \in \mathbf{Z}[z^{-1}, z]$ [3, 10] is an invariant of an isotopy type of an unoriented link L , which is defined by the following formulas:

$$Q(U; z) = 1; \quad (5.20)$$

$$Q(L_+; z) + Q(L_-; z) = z(Q(L_0; z) + Q(L_\infty; z)). \quad (5.21)$$

where U is the unknot and L_+ , L_- , L_0 , L_∞ are four unoriented links that are identical except near one point where they are as in Figure 10. We call $(L_+, L_-, L_0, L_\infty)$ an *unoriented skein quadruple*.

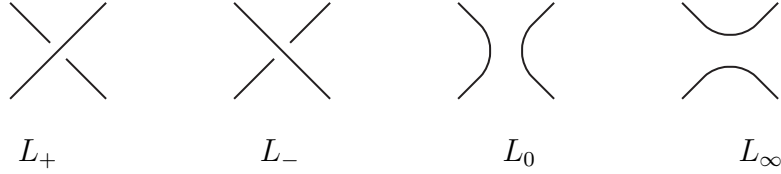


FIGURE 10. An unoriented skein quadruple.

Put $\rho(L) = Q(L; (\sqrt{5} - 1)/2)$. For a link L , Jones [13] has shown:

$$\rho(L) = \pm\sqrt{5}^r, \quad (5.22)$$

where $r = \dim H_1(\Sigma_2(L); \mathbf{Z}_5)$. From the proof of [30, Theorem 2], we have the following:

Proposition 5.9. *If two links L and M are related by a single band surgery, then*

$$\rho(L)/\rho(M) \in \{\pm 1, \sqrt{5}^{\pm 1}\}. \quad (5.23)$$

This immediately implies the following:

Corollary 5.10. *Let L and M be links.*

- (i) *If $\rho(L)/\rho(M) = \sqrt{5}^n$, then $d_b(L, M) \geq |n|$.*
- (ii) *If $\rho(L)/\rho(M) = -\sqrt{5}^n$, then $d_b(L, M) \geq |n| + 1$.*

Remark 5.11. Since $d_b(J, K) \leq d_2(J, K)$, Corollary 5.10 implies Corollary 8.1 in [17] and Corollary 8.2 in [16].

Example 5.12. (i) Let $K = 9_{49}$ or 10_{103} . Then $u_2(K) = 3$ is proved in [17, p. 453] by using $\rho(K) = -5$, which further implies $u_b(K) = 3$ by Corollary 5.10(ii).

(ii) Let F_n be the connected sum of n copies of the knot 5_1 . Since $\rho(5_1) = \sqrt{5}$ and $u_2(5_1) = u_b(5_1) = 1$, by Corollary 5.10(i) we have $u_2(F_n) = u_b(F_n) = n$. Since $\rho(4_1) = -\sqrt{5}$ and $u_2(4_1) = 2$, by Corollary 5.10(ii) we have $u_2(4_1 \# F_n) = u_b(4_1 \# F_n) = n + 2$. Note that $u_2(4_1 \# 5_1) = 3$ is given in [17, Table 9.3].

6. RELATIONS BETWEEN THE BAND- AND USUAL GORDIAN DISTANCES

We denote by $d(J, K)$ the usual Gordian distance between two knots J and K . Then we have the following; see [16, Theorem 10.1]:

$$d_2(J, K) \leq d(J, K) + 1. \quad (6.1)$$

Then by Theorem 3.1 we obtain $d_b(J, K) \leq d(J, K) + 1$. Moreover, we have the following:

Theorem 6.1. *For knots J and K , we have the following:*

$$d_b(J, K) \leq \begin{cases} d(J, K) & \text{if } d(J, K) \text{ is even;} \\ d(J, K) + 1 & \text{if } d(J, K) \text{ is odd.} \end{cases} \quad (6.2)$$

In particular,

$$u_b(K) \leq \begin{cases} u(K) & \text{if } u(K) \text{ is even;} \\ u(K) + 1 & \text{if } u(K) \text{ is odd.} \end{cases} \quad (6.3)$$

Also, if J and K are unknotting number one knots, then

$$d_b(J, K) \leq 2; \quad (6.4)$$

$$u_b(J \# K) \leq 2. \quad (6.5)$$

We prove this using the following Lemma.

Lemma 6.2. *Suppose that two knots J and K are related by a crossing change, and let H be the Hopf link. Then*

$$d_b(J, K \# H) = 1. \quad (6.6)$$

In particular, if K is an unknotting number one knot. Then we have:

$$d_b(K, H) = u_b(K \# H) = 1. \quad (6.7)$$

Proof. Suppose that J is transformed into K by changing the crossing as shown in Fig. 11(a). Attaching a band as shown Fig. 11(b) to J , we obtain $K \# H$ as shown in Figs. 11(c) or (d). Thus we obtain Eq. (6.6). \square

Proof of Theorem 6.1. If $n = 1$, then $d_b(J, K) \leq d_b(J, K \# H) + d_b(K \# H, K) \leq 2$, where we use Eq. (6.7) and the fact that H is transformed into the trivial knot by a single band surgery. Let $n > 1$ and let $J = J_0, J_1, \dots, J_n = K$ be a sequence of knots such that J_{k+1} is obtained from J_k by a crossing change. Then by Lemma 6.2, J_k and $J_{k+1} \# H$ are related by a band surgery, and so $d_b(J_k, J_{k+2}) \leq 2$. Thus we obtain the result. \square

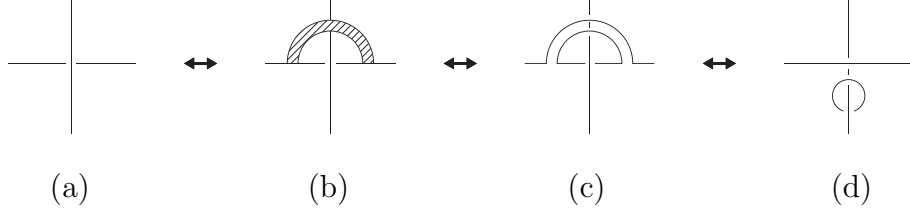


FIGURE 11

Example 6.3. Since 6_1 and 7_7 have unknotting number one, by Eq. (6.4) in Theorem 6.1 we have $d_b(6_1!, 7_7) \leq 2$. In [16, Table 5] it is listed that $d_2(6_1!, 7_7) = 2$ or 3, and so by Theorem 3.1 we obtain $d_b(6_1!, 7_7) = 2$.

Proposition 6.4. *If J and K are knots with $d_2(J, K) = 3$ and $u_b(J, K) = 2$, then $|\sigma(J) - \sigma(K)| \leq 2$.*

Proof. By the conditions there exists a 2-component link L such that J and L are related by an oriented band surgery and that L and K are related by an oriented band surgery. Then by Eq. (5.17) we obtain $|\sigma(J) - \sigma(L)| \leq 1$ and $|\sigma(L) - \sigma(K)| \leq 1$, which implies the result. \square

In [16, Example 6.4], several examples of knots with $H(2)$ -unknotting number 3 and pairs of knots with $H(2)$ -Gordian distance 3, which were proved by applying Proposition 7.2 below. In the following, we determine the band-unknotting numbers and band-Gordian distances for these examples.

Example 6.5. (i) In [16, Example 6.4(i)] $u_2(K) = 3$ is proved for $K = 8_{18}, 3_1! \# 8_{21}, 3_1 \# 9_{40}, 6_2 \# 9_{35}$. Thus by Theorem 3.1 $u_b(K) = 2$ or 3. Moreover, we have:

$$u_b(8_{18}) = u_b(3_1! \# 8_{21}) = 2; \quad u_b(3_1 \# 9_{40}) = u_b(6_2 \# 9_{35}) = 3. \quad (6.8)$$

Proof. Since $u(8_{18}) = 2$, by Theorem 6.1 we have $u_b(8_{18}) \leq 2$, and so we obtain $u_b(8_{18}) = 2$. Also, attaching the band to the knot 8_{18} as shown in Figure 12(a), we obtain the composite link $3_1 \# H$, which has band-unknotting number one by Eq. (6.7) in Lemma 6.2, and so $u_b(8_{18}) \leq 2$.

Since the knots $3_1!, 8_{21}$ have unknotting number one, by Eq. (6.5) in Theorem 6.1 we have $u_b(3_1! \# 8_{21}) \leq 2$, and so $u_b(3_1! \# 8_{21}) = 2$.

Since $\rho(3_1 \# 9_{40}) = -5$, by Corollary 5.10(ii) we have $u_b(3_1 \# 9_{40}) \geq 3$, and so $u_b(3_1 \# 9_{40}) = 3$.

Lastly, if $u_b(6_2 \# 9_{35}) = 2$, then by Proposition 6.4 $|\sigma(6_2 \# 9_{35})| \leq 2$. This is a contradiction since $\sigma(6_2) = \sigma(9_{35}) = 2$, and so $\sigma(6_2 \# 9_{35}) = 4$. Therefore, we obtain $u_b(6_2 \# 9_{35}) = 3$. Notice that since $\sigma(3_1 \# 9_{40}) = 4$, we may prove $u_b(3_1 \# 9_{40}) = 3$ in the same way. \square

(ii) In [16, Example 6.4(ii)], $d_2(5_1, 3_1 \# 3_1) = d_2(5_1!, 3_1 \# 3_1) = d_2(6_3, 3_1 \# 3_1!) = 3$ are proved. Thus by Theorem 3.1 the band-Gordian distance of these pairs are either 2 or 3. Moreover, we have:

$$d_b(5_1, 3_1 \# 3_1) = d_b(6_3, 3_1 \# 3_1!) = 2; \quad d_b(5_1!, 3_1 \# 3_1) = 3. \quad (6.9)$$

Proof. Attaching the band to the knot 5_1 as shown in Fig. 12(b), we obtain the composite link $3_1 \# H$. Since $d_b(3_1, H) = 1$ by Eq. (6.7) in Lemma 6.2, we have $d_b(3_1 \# H, 3_1 \# 3_1) = 1$, and so $d_b(5_1, 3_1 \# 3_1) \leq 2$. This also follows from the fact that $d(5_1, 3_1 \# 3_1) = 2$ by using Theorem 6.1; see [5]. Thus we obtain $d_b(5_1, 3_1 \# 3_1) = 2$.

Attaching the band to the knot 6_3 as shown in Fig. 12(c), we obtain the composite link $3_1 \# H$. Since $d_b(3_1!, H) = 1$ by Eq. (6.6) in Lemma 6.2, we have $d_b(3_1 \# H, 3_1 \# 3_1!) = 1$, and so $d_b(6_3, 3_1 \# 3_1!) \leq 2$. Note that $d(6_3, 3_1 \# 3_1!) = 2$; see [5]. Thus we obtain $d_b(6_3, 3_1 \# 3_1!) = 2$.

We may prove $d_b(5_1!, 3_1 \# 3_1) = 3$ by Proposition 6.4 since $\sigma(5_1!) = -4$ and $\sigma(3_1 \# 3_1) = 4$. \square

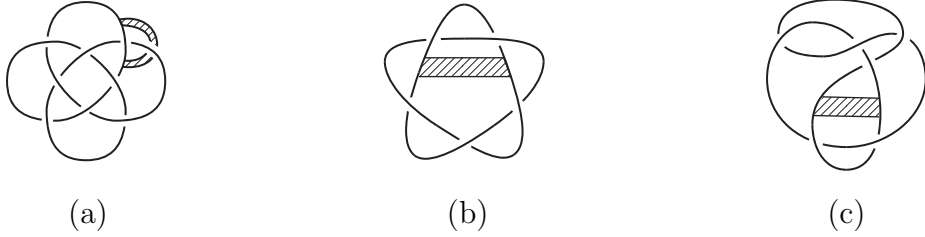


FIGURE 12

7. KNOTS WITH $u_b = 2$ AND $u_2 = 3$

In this section, we prove the following theorem.

Theorem 7.1. *There exist infinitely many knots K such that $u_b(K) = 2$ and $u_2(K) = 3$.*

We use the following proposition [16, Theorem 6.3].

Proposition 7.2. *Let J and K be knots with $V(J; \omega)/V(K; \omega) = 3$. If*

- (i) $\sigma(J) - \sigma(K) \equiv 0 \pmod{8}$, $\text{Arf}(J) \neq \text{Arf}(K)$, or
- (ii) $\sigma(J) - \sigma(K) \equiv 4 \pmod{8}$, $\text{Arf}(J) = \text{Arf}(K)$,

then $d_2(J, K) \geq 3$.

Let K_m be a knot as shown in Figure 13, where the tangle labeled m stands for a 2-braid with $|m|$ crossings in the manner indicated in Fig. 14. Then we see that $K_0 = 6_3$, $K_{-1} = 6_2$, $K_1 = 8_{21}$, $K_{-2} = 8_{20!}$, $K_2 = 9_{44}$, $K_{-3} = 9_{42!}$, $K_3 = 10_{133}$, $K_{-4} = 10_{132!}$.

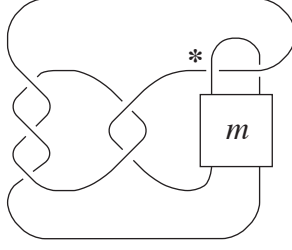


FIGURE 13

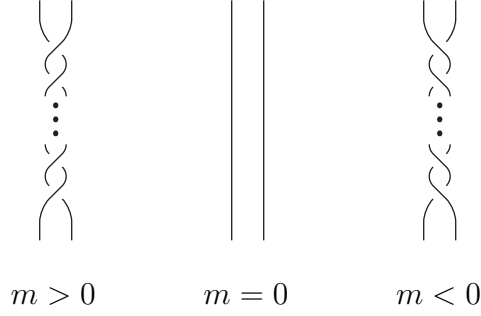


FIGURE 14

We have a skein triple (K_{m-2}, K_m, H) , where H is a positive or negative Hopf link according as if m is even or odd. Then from Eq. (5.7), we obtain the following:

$$\begin{aligned}
 t^{-1}V(K_{m-2}; t) - tV(K_m; t) &= (t^{1/2} - t^{-1/2}) V(H; t) \\
 &= \begin{cases} 1 - t + t^2 - t^3 & \text{if } m \text{ is even;} \\ t^{-3} - t^{-2} + t^{-1} - 1 & \text{if } m \text{ is odd;} \end{cases} \quad (7.1)
 \end{aligned}$$

and

$$V(K_{-1}; t) = V(6_2; t) = t^{-5} - 2t^{-4} + 2t^{-3} - 2t^{-2} + 2t^{-1} - 1 + t; \quad (7.2)$$

$$V(K_0; t) = V(6_3; t) = -t^{-3} + 2t^{-2} - 2t^{-1} + 3 - 2t + 2t^2 - t^3. \quad (7.3)$$

Using these inequalities, we prove the following three lemmas.

Lemma 7.3.

$$\text{Arf}(K_{2n}) = \text{Arf}(K_{2n-1}) \equiv n + 1 \pmod{2}. \quad (7.4)$$

Proof. Putting $t = i$ in Eqs. (7.1), (7.2) and (7.3), we have $V(K_{n-2}; i) + V(K_n; i) = 0$ and $V(K_{-1}; i) = V(K_0; i) = -1$. Then using Eq. (5.16), we have $\text{Arf}(K_{n-2}) \neq \text{Arf}(K_n)$ and $\text{Arf}(K_0) = \text{Arf}(K_{-1}) = 1$, which imply the result. \square

Lemma 7.4.

$$\sigma(K_{2n}) = \begin{cases} 0 & \text{if } n \geq -3; \\ -2 & \text{if } n \leq -4; \end{cases} \quad \sigma(K_{2n-1}) = \begin{cases} 0 & \text{if } n \leq -3; \\ 2 & \text{if } n \geq -2. \end{cases} \quad (7.5)$$

Proof. The unknotting number of K_n is one. Indeed, by changing the crossing near the mark $*$ indicated in Figure 13, K_n becomes the trivial knot. Thus by Eq. (5.18) and the fact that the signature of a knot is even, we have

$$\sigma(K_n) = 0 \text{ or } \pm 2. \quad (7.6)$$

Furthermore, by Eq. (5.18), we have

$$\sigma(K_{n-2}) \leq \sigma(K_n). \quad (7.7)$$

Putting $t = -1$ in Eqs. (7.1), (7.2) and (7.3), we have

$$V(K_n; -1) - V(K_{n-2}; -1) = \begin{cases} -4 & \text{if } n \text{ is even;} \\ 4 & \text{if } n \text{ is odd;} \end{cases} \quad (7.8)$$

$$V(K_{-1}; -1) = -11 \text{ and } V(K_0; -1) = 13.$$

Thus we obtain the following:

$$\text{sign}V(K_{2n}; -1) = \begin{cases} 1 & \text{if } n \geq -3; \\ -1 & \text{if } n \leq -4; \end{cases} \quad (7.9)$$

$$\text{sign}V(K_{2n-1}; -1) = \begin{cases} 1 & \text{if } n \leq -3; \\ -1 & \text{if } n \geq -2. \end{cases} \quad (7.10)$$

Therefore, $\sigma(K_{2n})/2$ is even or odd according as if $n \geq -3$ or $n \leq -4$; and $\sigma(K_{2n-1})/2$ is even or odd according as if $n \leq -3$ or $n \geq -2$. Then using Eq. (7.7), we obtain the result. \square

Lemma 7.5.

$$V(K_{2n}; \omega) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3}; \\ -1 & \text{if } n \equiv 1 \pmod{3}; \\ i\sqrt{3} & \text{if } n \equiv 2 \pmod{3}; \end{cases} \quad (7.11)$$

$$V(K_{2n-1}; \omega) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3}; \\ -i\sqrt{3} & \text{if } n \equiv 1 \pmod{3}; \\ -1 & \text{if } n \equiv 2 \pmod{3}. \end{cases} \quad (7.12)$$

Proof. Putting $t = \omega$ in Eqs. (7.1), (7.2), and (7.3), we have:

$$V(K_n; \omega) + \omega V(K_{n-2}; \omega) = \begin{cases} \omega^2 & \text{if } n \text{ is even;} \\ -\omega^2 & \text{if } n \text{ is odd;} \end{cases} \quad (7.13)$$

$$V(K_{-1}; \omega) = V(K_0; \omega) = 1.$$

Using these inequalities, we obtain the result. \square

Proof of Theorem 7.1. Let

$$J_l = \begin{cases} K_{12l+1} \# 3_1! & \text{if } l \geq 0; \\ K_{12l-2} \# 3_1 & \text{if } l \leq -1. \end{cases} \quad (7.14)$$

For each $l \in \mathbf{Z}$ we prove:

$$u_b(J_l) = 2, \quad u_2(J_l) = 3. \quad (7.15)$$

By lemmas 7.3, 7.4 and 7.5 we obtain Table 1, from which we have the following:

$$\sigma(J_l) = 0, \quad \text{Arf}(J_l) = 1, \quad V(J_l; \omega) = 3, \quad (7.16)$$

and so by Proposition 7.2, we have $u_2(J_l) \geq 3$. On the other hand, since K_m and 3_1 have unknotting number one, by Theorem 6.1 we have $u_b(J_l) \leq 2$. Therefore, by Theorem 3.1, we obtain $u_b(J_l) = 2$ and $u_2(J_l) = 3$. This completes the proof. \square

TABLE 1. Unknotting number one knots with $V(\omega) = \pm i\sqrt{3}$.

Knots K	K_{12l+1} ($l \geq 0$)	K_{12l+4} ($l \geq 0$)	K_{12l+7} ($l \leq -1$)	K_{12l-2} ($l \leq -1$)	3_1	$3_1!$	6_1	$6_1!$
$\sigma(K)$	2	0	0	-2	2	-2	0	0
$\text{Arf}(K)$	0	1	1	0	1	1	0	0
$V(K; \omega)$	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$	$i\sqrt{3}$	$i\sqrt{3}$	$-i\sqrt{3}$

Remark 7.6. Note that $J_0 = 8_{21} \# 3_1!$ and $u_2(J_0) = 3$ are given in Example 6.4(i) in [16].

Remark 7.7. Let

$$\tilde{J}_l = \begin{cases} K_{12l+4} \# 6_1! & \text{if } l \geq 0; \\ K_{12l+7} \# 6_1 & \text{if } l \leq -1. \end{cases} \quad (7.17)$$

Then from Table 1 we have $\sigma(\tilde{J}_l) = 0$, $\text{Arf}(\tilde{J}_l) = 1$, $V(\tilde{J}_l; \omega) = 3$, and since 6_1 has unknotting number one, we obtain $u_b(\tilde{J}_l) = 2$ and $u_2(\tilde{J}_l) = 3$ for each integer l .

Remark 7.8. Yasuhara [34] has proved that if a knot K bounds a Möbius band in B^4 such that $\partial B^4 = S^3$, then

$$\sigma(K) - 4\text{Arf}(K) \equiv 0 \text{ or } \pm 2 \pmod{8}. \quad (7.18)$$

If a knot K is transformed into the unknot by a band surgery, then K bounds a Möbius band in B^4 , and so K satisfies Eq. (7.18), which was generalized in [16], and Proposition 7.2 is a related result. Another generalization was given by Gilmer and Livingston [9].

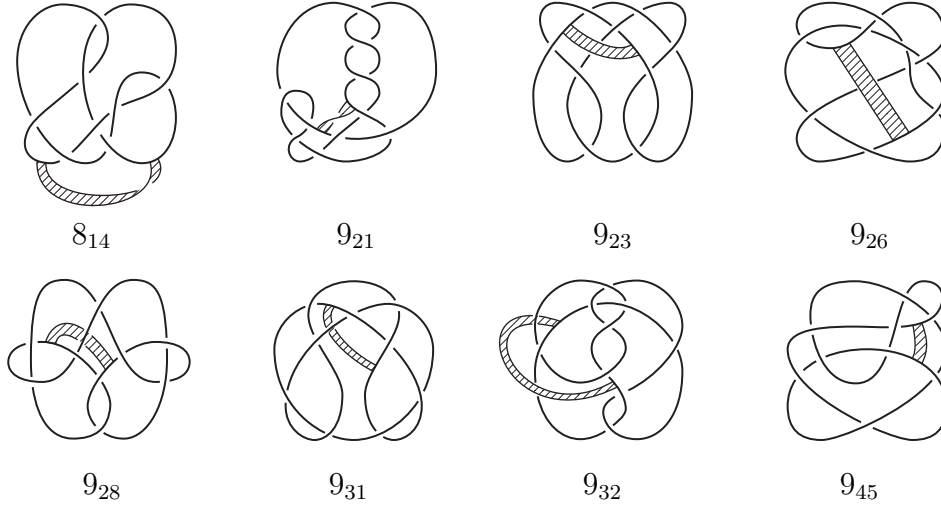
Remark 7.9. Let $C(K)$ be the *crosscap number* of a knot K ; see [4]. Then $d_2(K) \leq C(K)$. Note that $d_b(K) \leq C(K)$ is given in [32, Proposition 5.3].

8. BAND-UNKNOTTING NUMBERS AND $H(2)$ -UNKNOTTING NUMBERS OF KNOTS WITH UP TO 9 CROSSINGS

Table 2 lists the band-unknotting numbers, $H(2)$ -unknotting numbers, and usual unknotting numbers of knots with up to 9 crossings. In [17] a table of the $H(2)$ -unknotting numbers of knots with up to 9 crossings is given, but there were 8 knots whose $H(2)$ -unknotting numbers were undecided. This table is the complete list for the $H(2)$ -unknotting numbers of knots with up to 9 crossings: In [16, Example 6.4] $u_2(8_{18}) = 3$ is proved, and Bao [1] has given a condition for a 2-bridge knot to have $H(2)$ -unknotting number one, which implies the 2-bridge knots 9_{21} , 9_{23} , 9_{26} , 9_{31} have $H(2)$ -unknotting number one. We show these knots and the remaining knots 9_{28} , 9_{32} , 9_{45} have $H(2)$ -unknotting number one in Fig. 15 giving twisted bands that transform into the unknot.

TABLE 2. Band-unknotting numbers, $H(2)$ -unknotting numbers, and unknotting numbers of knots with up to 9 crossings.

u_b	u_2	u	knots
1	1	1	$3_1, 6_1, 6_2, 7_2, 7_6, 8_7, 8_{11}, 8_{14}, 8_{20}, 9_{19}, 9_{21}, 9_{22}, 9_{26-9_{28}}, 9_{42}, 9_{44}, 9_{45}$.
1	1	2	$5_1, 7_3, 7_4, 8_3-8_6, 8_8, 8_{10}, 8_{16}, 9_4, 9_5, 9_7, 9_8, 9_{15}, 9_{17}, 9_{24}, 9_{29}, 9_{31}, 9_{32}, 9_{36}, 9_{43}$.
1	1	3	$7_1, 8_{19}, 9_3, 9_6, 9_9, 9_{13}$.
1	1	4	9_1 .
2	2	1	$4_1, 6_3, 7_7, 8_1, 8_9, 8_{13}, 8_{17}, 8_{21}, 9_2, 9_{12}, 9_{14}, 9_{24}, 9_{30}, 9_{33}, 9_{34}, 9_{39}$.
2	2	2	$7_5, 8_2, 8_{12}, 8_{15}, 9_{11}, 9_{18}, 9_{20}, 9_{37}, 9_{40}, 9_{41}, 9_{46-9_{48}},$ $3_1\#3_1, 3_1!\#3_1, 3_1!\#5_2, 4_1\#4_1, 3_1\#6_1, 3_1!\#6_1, 3_1\#6_2, 3_1\#6_3$.
2	2	3	$9_{10}, 9_{16}, 9_{35}, 9_{38}, 3_1\#5_1$.
2	2	2-3	$3_1!\#5_1$.
2	3	2	8_{18} .
3	3	3	$9_{49}, 3_1\#3_1\#3_1, 3_1!\#3_1\#3_1, 4_1\#5_1$.

FIGURE 15. $H(2)$ -unknotting number one knots.

Remark 8.1. In [17, Fig. 9.1] the indicated twisted band does not transform the knot 8_{14} into the unknot. The corrected band is shown in Fig. 15. In [17, Table 9.3] there are errors in the unknotting numbers; the right ones are: $u(3_1 \# 5_1) = 3$, $u(3_1! \# 5_1) = 2$ or 3 .

9. SPECIAL VALUE OF THE JONES POLYNOMIAL AND A SLICE KNOT

A knot in S^3 is *slice* if it bounds a disk in the 4-ball B^4 such that $\partial B^4 = S^3$. Using an oriented version of Theorem 5.5 we have proved [15, Corollary 4.5]: If K is a ribbon knot, then $V(K; \omega) \neq -1$. Moreover, we have:

Proposition 9.1. *If K is a slice knot, then $V(K; \omega) \neq -1$.*

Proof. Let K be a slice knot with $V(K; \omega) = -1$. First, note that by Eq. (5.8) the determinant of K is not divided by 3. There are polynomial $P(t)$ in $\mathbf{Z}[t^{-1}, t]$ and integers a, b such that

$$V(K; t) = (t^2 - t + 1)P(t) + at + b. \quad (9.1)$$

Then we have $a = 0$ and $b = -1$, and so we obtain:

$$V(K; -1) = 3P(-1) - 1. \quad (9.2)$$

On the other hand, the value $|V(K; -1)|$ is the determinant of K , which is a square integer. More precisely, $V(K; -1) = \Delta_K(-1)$, the value of the Conway-normalized Alexander polynomial $\Delta_K(t)$ at $t = -1$, where $\Delta_K(t)$ satisfies $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$; see [21, Chapter 16]. Since K is a slice knot, $\Delta_K(t)$ has the form $\Delta_K(t) = f(t)f(t^{-1})$, where $f(t)$ is a polynomial in $\mathbf{Z}[t]$ [7]; cf. [29, Theorem 8E20]. Putting $f(-1) = 3k \pm 1$, $V(K; -1) = 9k^2 \pm 6k + 1$, which contradicts Eq. (9.2). This completes the proof. \square

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